K_1-injectivity for properly infinite C*-algebras

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K_1-INJECTIVITY FOR PROPERLY INFINITE C*-ALGEBRAS

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Dedicated to Alain Connes on the occasion of his 60th birthday.

1. Introduction

One of the main tools to classify C*-algebras is the study of its projections and its unitaries. It was proved by J. Cuntz in [Cun81] that if A is a *purely infinite* simple C*-algebra, then the kernel of the natural map for the unitary group $\mathcal{U}(A)$ to the K-theory group $K_1(A)$ is reduced to the connected component $\mathcal{U}^0(A)$, *i.e.* A is K_1*-injective* (see §3). We study in this note a finitely generated C*-algebra, the K_1-injectivity of which would imply the K_1-injectivity of all unital *properly infinite* C*-algebras.

Note that such a question was already considered in [Blac07], [BRR08].

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2. Preliminaries

Let us first review briefly the theory introduced by J. Cuntz ([Cun78]) of comparison of positive elements in a C*-algebra.

Definition 2.1. ([Cun78], [Rør92]) Given two positive elements a, b in a C*-algebra A, one says that:
– a is *dominated* by b (written $a \precsim b$) if and only if there is a sequence $\{d_k; k \in \mathbb{N}\}$ in A such that $\|d_k^* b d_k - a\| \to 0$ when $k \to \infty$,
– a is *properly infinite* if $a \neq 0$ and $a \oplus a \precsim a \oplus 0$ in the C*-algebra $M_2(A) := M_2(\mathbb{C}) \otimes A$.

This leads to the following notions of infiniteness for C*-algebras.

Definition 2.2. ([Cun78], [Cun81], [KR00]) A unital C*-algebra A is said to be:
– *properly infinite* if its unit 1_A is properly infinite in A,
– *purely infinite* if all the non zero positive elements in A are properly infinite in A.

Remark 2.3. E. Kirchberg and M. Rørdam have proved in [KR00, Theorem 4.16] that a C*-algebra A is purely infinite (in the above sense) if and only if there is no character on the C*-algebra A and any positive element a in A which lies in the closed two-sided ideal generated by another positive element b in A satisfies $a \precsim b$.

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The first examples of such C*-algebras were given by J. Cuntz in [Cun81]: For any integer $n \geq 2$, the C*-algebra $\mathcal{T}_n$ is the universal unital C*-algebra generated by n isometries $s_1, \ldots, s_n$ satisfying the relation

$$s_1 s_1^* + \ldots + s_n s_n^* \leq 1 \tag{2.1}$$

Then, the closed two sided ideal in $\mathcal{T}_n$ generated by the *minimal* projection $p_{n+1} := 1 - s_1 s_1^* - \ldots - s_n s_n^*$ is isomorphic to the C*-algebra $\mathcal{K}$ of compact operators on an infinite dimension separable Hilbert space and one has an exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}_n \xrightarrow{\pi} \mathcal{O}_n \to 0\,, \tag{2.2}$$

where the quotient $\mathcal{O}_n$ is a purely infinite *simple* unital nuclear C*-algebra ([Cun81]).

Remark 2.4. A unital C*-algebra A is properly infinite if and only if there exists a unital $*$-homomorphism $\mathcal{T}_2 \to A$.

3. K_1-injectivity of $\mathcal{T}_n$

Given a unital C*-algebra A with unitary group $\mathcal{U}(A)$, denote by $\mathcal{U}^0(A)$ the connected component of 1_A in $\mathcal{U}(A)$. For each strictly positive integer $k \geq 1$, the upper diagonal embedding $u \in \mathcal{U}(M_k(A)) \mapsto (u \oplus 1_A) \in \mathcal{U}(M_{k+1}(A))$ sends the connected component $\mathcal{U}^0(M_k(A))$ into $\mathcal{U}^0(M_{k+1}(A))$, whence a canonical homomorphism Θ_A from $\mathcal{U}(A)/\mathcal{U}^0(A)$ to $K_1(A) := \lim\limits_{k\to\infty} \mathcal{U}(M_k(A))/\mathcal{U}^0(M_k(A))$. As noticed by B. Blackadar in [Blac07], this map is (1) neither injective, (2) nor surjective in general:

(1) If $\mathfrak{U}_2$ denotes the compact unitary group of the matrix C*-algebra $M_2(\mathbb{C})$, $A := C(\mathfrak{U}_2 \times \mathfrak{U}_2, M_2(\mathbb{C}))$ and $u, v \in \mathcal{U}(A)$ are the two unitaries $u(x,y) = x$ and $v(x,y) = y$, then $z := uvu^*v^*$ is not unitarily homotopic to 1_A in $\mathcal{U}(A)$ but the unitary $z \oplus 1_A$ belongs to $\mathcal{U}^0(M_2(A))$ ([AJT60]).
(2) If $A = C(\mathbb{T}^3)$, then $\mathcal{U}(A)/\mathcal{U}^0(A) \cong \mathbb{Z}^3$ but $K_1(A) \cong \mathbb{Z}^4$.

Definition 3.1. The unital C*-algebra A is said to be K_1-*injective* if the map Θ_A is injective.

J. Cuntz proved in [Cun81] that Θ_A is surjective as soon as the C*-algebra A is properly infinite and that it is also injective if the C*-algebra A is simple and purely infinite. Now, the K-theoretical six-term cyclic exact sequence associated to the exact sequence (2.2) implies that $K_1(\mathcal{T}_n) = 0$ since $K_1(\mathcal{K}) = K_1(\mathcal{O}_n) = 0$. Thus, the map $\Theta_{\mathcal{T}_n}$ is zero.

Proposition 3.2. *For all $n \geq 2$, the C*-algebra $\mathcal{T}_n$ is K_1-injective, i.e. any unitary $u \in \mathcal{U}(\mathcal{T}_n)$ is unitarily homotopic to $1_{\mathcal{T}_n}$ in $\mathcal{U}(\mathcal{T}_n)$ (written $u \sim_h 1_{\mathcal{T}_n}$).*

Proof. The C*-algebras $\mathcal{T}_n$ have real rank zero by Proposition 2.3 of [Zha90]. And Lin proved that any unital C*-algebra of real rank zero is K_1-injective ([Lin01, Corollary 4.2.10]). □

Corollary 3.3. *If $\alpha : \mathcal{T}_3 \to \mathcal{T}_3$ is a unital $*$-endomorphism, then its restriction to the unital copy of $\mathcal{T}_2$ generated by the two isometries s_1, s_2 is unitarily homotopic to $id_{\mathcal{T}_2}$ among all unital $*$-homomorphisms $\mathcal{T}_2 \to \mathcal{T}_3$.*

Proof. The isometry $\sum_{k=1,2}\alpha(s_k)s_k^*$ extends to a unitary $u\in\mathcal{U}(\mathcal{T}_3)$ such that $\alpha(s_k)=us_k$ for $k=1,2$ ([BRR08, Lemma 2.4]). But Proposition 3.2 yields that $\mathcal{U}(\mathcal{T}_3)=\mathcal{U}^0(\mathcal{T}_3)$, whence a homotopy $u\sim_h 1$ in $\mathcal{U}(\mathcal{T}_3)$, and so the desired result holds. $\square$

Remark 3.4. The unital map $\iota:\mathbb{C}\to\mathcal{T}_2$ induces an isomorphism in K-theory. Indeed, $[1_{\mathcal{T}_2}]=[s_1s_1^*]+[s_2s_2^*]+[p_3]=2\,[1_{\mathcal{T}_2}]+[p_3]$ and so $[1_{\mathcal{T}_2}]=-[p_3]$ is invertible in $K_0(\mathcal{T}_2)$.

4. K_1-injectivity of properly infinite C*-algebras

Denote by $\mathcal{T}_2*_{\mathbb{C}}\mathcal{T}_2$ the universal unital free product with amalgamation over $\mathbb{C}$ (in the sequel called full unital free product) of two copies of $\mathcal{T}_2$ amalgamated over $\mathbb{C}$ and let $\jmath_0$, $\jmath_1$ be the two canonical unital inclusions of $\mathcal{T}_2$ in $\mathcal{T}_2*_{\mathbb{C}}\mathcal{T}_2$. We show in this section that the K_1-injectivity of $\mathcal{T}_2*_{\mathbb{C}}\mathcal{T}_2$ is equivalent to the K_1-injectivity of all the unital properly infinite C*-algebras. The proof is similar to that of the universality of the full unital free product $\mathcal{O}_\infty*_{\mathbb{C}}\mathcal{O}_\infty$ (see Theorem 5.5 of [BRR08]).

Definition 4.1. ([Blan09], [BRR08, §2]) If X is a compact Hausdorff space, a unital $C(X)$-algebra is a unital C*-algebra A endowed with a unital $*$-homomorphism from the C*-algebra $C(X)$ of continuous functions on X to the centre of A.

For any non-empty closed subset Y of X, we denote by π_Y^A (or simply by π_Y if no confusion is possible) the quotient map from A to the quotient A_Y of A by the (closed) ideal $C_0(X\setminus Y)\cdot A$. For any point $x\in X$, we also denote by A_x the quotient $A_{\{x\}}$ and by π_x the quotient map $\pi_{\{x\}}$.

Proposition 4.2. *The following assertions are equivalent.*

(i) $\mathcal{T}_2*_{\mathbb{C}}\mathcal{T}_2$ *is K_1-injective.*
(ii) $\mathcal{D}:=\{f\in C([0,1],\mathcal{T}_2*_{\mathbb{C}}\mathcal{T}_2)\,;\ f(0)\in\jmath_0(\mathcal{T}_2)\text{ and } f(1)\in\jmath_1(\mathcal{T}_2)\}$ *is properly infinite.*
(iii) *There exists a unital $*$-homomorphism $\theta:\mathcal{T}_2\to\mathcal{D}$.*
(iv) *There exists a projection $q\in\mathcal{D}$ with $\pi_0(q)=\jmath_0(s_1s_1^*)$ and $\pi_1(q)=\jmath_1(s_1s_1^*)$.*
(v) *Any unital properly infinite C*-algebra A is K_1-injective.*

Proof. (i)⇒(ii) We have a pull-back diagram

$$\begin{tikzcd} & \mathcal{D} \arrow[dl] \arrow[dr] & \\ \mathcal{D}_{[0,\frac12]} \arrow[dr, "\pi_{\frac12}"'] & & \mathcal{D}_{[\frac12,1]} \arrow[dl, "\pi_{\frac12}"] \\ & \mathcal{T}_2*_{\mathbb{C}}\mathcal{T}_2 & \end{tikzcd}$$

and the two C*-algebras $\mathcal{D}_{[0,\frac12]}$ and $\mathcal{D}_{[\frac12,1]}$ are properly infinite (Remark 2.4). Hence, the implication follows from [BRR08, Proposition 2.7].

(ii)⇒(iii) is Remark 2.4 applied to the C*-algebra $\mathcal{D}$.

(iii)⇒(iv) The two full, properly infinite projections $\jmath_0(s_1s_1^*)$ and $\pi_0\circ\theta(s_1s_1^*)$ are unitarily equivalent in $\jmath_0(\mathcal{T}_2)$ by [LLR00, Lemma 2.2.2] and [BRR08, Proposition 2.3].

Thus, they are homotopic among the projections in the C*-algebra $\jmath_0(\mathcal{T}_2)$ (written $\jmath_0(s_1s_1^*) \sim_h \pi_0 \circ \theta(s_1s_1^*)$) by Proposition 3.2. Similarly, $\pi_1 \circ \theta(s_1s_1^*) \sim_h \jmath_1(s_1s_1^*)$ in $\jmath_1(\mathcal{T}_2)$. Further, $\pi_0 \circ \theta(s_1s_1^*) \sim_h \pi_1 \circ \theta(s_1s_1^*)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ by hypothesis, whence the result by composition.

(iv)⇒(v) By [BRR08, Proposition 5.1], it is equivalent to prove that if p and p' are two properly infinite full projections in A, then there exist full properly infinite projections p_0, and p_0' in A such that $p_0 \leq p$, $p_0' \leq p'$ and $p_0 \sim_h p_0'$.

Fix two such projections p and p' in A. Then, there exist unital $*$-homomorphisms $\sigma : \mathcal{T}_2 \to pAp$, $\sigma' : \mathcal{T}_2 \to p'Ap'$ and isometries $t, t' \in A$ such that $1_A = t^*pt = (t')^*p't'$. Now, one thoroughly defines unital $*$-homomorphisms $\alpha_0 : \mathcal{T}_2 \to A$ and $\alpha_1 : \mathcal{T}_2 \to A$ by

$$\alpha_0(s_k) := \sigma(s_k) \cdot t \quad \text{and} \quad \alpha_1(s_k) := \sigma'(s_k) \cdot t' \qquad \text{for } k = 1, 2\,,$$

whence a unital $*$-homomorphism $\alpha := \alpha_0 * \alpha_1 : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \to A$ such that $\alpha \circ \jmath_0 = \alpha_0$ and $\alpha \circ \jmath_1 = \alpha_1$.

The two full properly infinite projections $p_0 = \alpha_0(s_1s_1^*)$ and $p_0' = \alpha_1(s_1s_1^*)$ satisfy $p_0 \leq p$ and $p_0' \leq p'$. Further, the projection $(id \otimes \alpha)(q)$ gives a continuous path of projections in A from p_0 to p_0'. ☐

Remark 4.3. The C*-algebra $M_2(\mathcal{D})$ is properly infinite by [BRR08, Proposition 2.7].

Lemma 4.4. *$K_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = \mathbb{Z}$ and $K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = 0$*

Proof. The commutative diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\imath_1"] \arrow[d, "\imath_0"'] & \mathcal{T}_2 \arrow[d, "\jmath_1"] \\ \mathcal{T}_2 \arrow[r, "\jmath_0"] & \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \end{tikzcd}$$

yields by [Ger97, Theorem 2.2] a six-term cyclic exact sequence

$$\begin{array}{ccccc} K_0(\mathbb{C}) = \mathbb{Z} & \overset{(\imath_0 \oplus \imath_1)_*}{\longrightarrow} & K_0(\mathcal{T}_2 \oplus \mathcal{T}_2) = \mathbb{Z} \oplus \mathbb{Z} & \overset{(\jmath_0)_* - (\jmath_1)_*}{\longrightarrow} & K_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) & \longleftarrow & K_1(\mathcal{T}_2 \oplus \mathcal{T}_2) = 0 \oplus 0 & \longleftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

Now, Remark 3.4 implies that the map $(\imath_0 \oplus \imath_1)_*$ is injective, whence the equalities. ☐

Remark 4.5. G. Skandalis noticed that the C*-algebra $\mathcal{T}_2$ is KK-equivalent to $\mathbb{C}$ and so $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is KK-equivalent to $\mathbb{C} *_{\mathbb{C}} \mathbb{C} = \mathbb{C}$.

This Lemma entails that the K_1-injectivity question for unital properly infinite C*-algebras boils down to knowing whether $\mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = \mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$. Note that Proposition 3.2 already yields that $\mathcal{U}(\mathcal{T}_2) *_{\mathbb{T}} \mathcal{U}(\mathcal{T}_2) \subset \mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$.

But the following holds.

Proposition 4.6. *Set $p_3 = 1 - s_1s_1^* - s_2s_2^*$ in the Cuntz algebra $\mathcal{T}_2$ and let u be the canonical unitary generating $C^*(\mathbb{Z})$.*
(i) The relations $\jmath_0(s_k) \mapsto s_k$ and $\jmath_1(s_k) \mapsto u\, s_k$ ($k = 1, 2$) uniquely define a unital $*$-homomorphism $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \to \mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})$ which is injective but not K_1-surjective.

(ii) The two projections $\jmath_0(p_3)$ and $\jmath_1(p_3)$ satisfy $\jmath_1(p_3) \not\sim \jmath_0(p_3)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$.
(iii) There is no $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ such that $\jmath_1(s_1s_1^* + s_2s_2^*) = v\,\jmath_0(s_1s_1^* + s_2s_2^*)\,v^*$.
(iv) There is a unitary $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ such that $\jmath_1(s_1s_1^*) = v\,\jmath_0(s_1s_1^*)\,v^*$.

Proof. (i) The unital C*-subalgebra of $\mathcal{O}_3$ generated by the two isometries s_1 and s_2 is isomorphic to $\mathcal{T}_2$, whence a unital C*-embedding $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \subset \mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ ([ADEL04]). Let Φ be the $*$-homomorphism from $\mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ to the free product $\mathcal{O}_3 *_{\mathbb{C}} C^*(\mathbb{Z}) = C^*\big(s_1, s_2, s_3, u\big)$ fixed by the relations

$$\Phi(\jmath_0(s_k)) = s_k \quad \text{and} \quad \Phi(\jmath_1(s_k)) = u\,s_k \quad \text{for } k = 1, 2, 3$$

and let $\Psi : \mathcal{O}_3 *_{\mathbb{C}} C^*(\mathbb{Z}) \to \mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ be the only $*$-homomorphism such that

$$\Psi(u) = \sum_{l=1}^{3} \jmath_1(s_l)\jmath_0(s_l)^* \quad \text{and} \quad \Psi(s_k) = \jmath_0(s_k) \quad \text{for } k = 1, 2, 3.$$

For all $k = 1, 2, 3$, we have the equalities:
– $\Psi \circ \Phi(\jmath_0(s_k)) = \Psi(s_k) = \jmath_0(s_k)$,
– $\Psi \circ \Phi(\jmath_1(s_k)) = \Psi(us_k) = \jmath_1(s_k)$,
– $\Phi \circ \Psi(s_k) = \Phi(\jmath_0(s_k)) = s_k$.

Also, $\Psi(u)^*\Psi(u) = \sum_{l,l'} \jmath_0(s_{l'})\jmath_1(s_{l'})^*\jmath_1(s_l)\jmath_0(s_l)^* = 1_{\mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3} = \Psi(u)\Psi(u)^*$, *i.e.* $\Psi(u)$ is a unitary in $\mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ which satisfies:
– $\Phi \circ \Psi(u) = \sum_{l=1,2,3} \Phi(\jmath_1(s_l))\Phi(\jmath_0(s_l)^*) = u$.

Thus, Φ is an invertible unital $*$-homomorphism with inverse Ψ ([Blac07]), and the restriction of Φ to the C*-subalgebra $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ takes values in $\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z}) \subset \mathcal{O}_3 *_{\mathbb{C}} C^*(\mathbb{Z})$.

Now, there is (see [Ger97]) a six-term cyclic exact sequence

$$\begin{array}{ccccc}
K_0(\mathbb{C}) = \mathbb{Z} & \hookrightarrow & K_0\big(\mathcal{T}_2 \oplus C^*(\mathbb{Z})\big) = \mathbb{Z} \oplus \mathbb{Z} & \to & K_0(\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})\,) \\
\uparrow & & & & \downarrow \\
K_1(\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})\,) & \leftarrow & K_1\big(\mathcal{T}_2 \oplus C^*(\mathbb{Z})\big) = 0 \oplus \mathbb{Z} & \leftarrow & K_1(\mathbb{C}) = 0
\end{array}$$

and so, $K_1(\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})\,) = \mathbb{Z}$, whereas $K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = 0$ by Lemma 4.4.

(ii) Let $\pi_0 : \mathcal{T}_2 \to L(H)$ be a unital $*$-representation on a separable Hilbert space H such that $\pi_0(p_3)$ is a rank one projection, let $\pi_1 : \mathcal{T}_2 \to L(H)$ be a unital $*$-representation such that $\pi_1(p_3)$ is a rank two projection and consider the induced unital $*$-representation $\pi = \pi_0 * \pi_1$ of the unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$.

Then the two projections $\pi[\jmath_0(p_3)] = \pi_0(p_3)$ and $\pi[\jmath_1(p_3)] = \pi_1(p_3)$ have distinct ranks and so cannot be equivalent in $L(H)$. Hence, $\jmath_0(p_3) \not\sim \jmath_1(p_3)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$.

(iii) This is just a rewriting of the previous assertion since $s_1s_1^* + s_2s_2^* = 1 - p_3$. Indeed, the partial isometry $b = \jmath_1(s_1)\jmath_0(s_1)^* + \jmath_1(s_2)\jmath_0(s_2)^*$ defines a Murray-von Neumann equivalence in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ between the projections $\jmath_0(s_1s_1^* + s_2s_2^*) = 1 - \jmath_0(p_3)$ and $\jmath_1(s_1s_1^* + s_2s_2^*) = 1 - \jmath_1(p_3)$. Thus, they are unitarily equivalent in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ if and only if $\jmath_0(p_3) \sim \jmath_1(p_3)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ ([LLR00, Proposition 2.2.2]).

(iv) There exists a unitary $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ (which is necessarily K_1-trivial by Lemma 4.4) such that $\jmath_1(s_1s_1^*) = v\,\jmath_0(s_1s_1^*)\,v^*$. Indeed, we have the inequalities

$$1 > s_2s_2^* + p_3 > s_2s_2^* > s_2s_1(s_2s_2^* + p_3)s_1^*s_2^* + s_2s_2(s_2s_2^* + p_3)s_2^*s_2^* \qquad \text{in } \mathcal{T}_2\,.$$

Thus, if we set $w := \jmath_1(s_1)\jmath_0(s_1)^*$, then $1 - w^*w = \jmath_0(s_2s_2^* + p_3)$ and $1 - ww^* = \jmath_1(s_2s_2^* + p_3)$ are two properly infinite and full K_0-equivalent projections in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$. Thus, there is a partial isometry $a \in \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ with $a^*a = 1 - w^*w$ and $aa^* = 1 - ww^*$ ([Cun81]). The sum $v = a + w$ has the required properties ([BRR08, Lemma 2.4]). $\square$

Remarks 4.7. (i) The equivalence (iv)$\Leftrightarrow$(v) in Proposition 4.2 implies that all unital properly infinite C*-algebras are K_1-injective if and only if the unitary $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ constructed in Proposition 4.6.(iv) belongs to the connected component $\mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$.

Note that $v \oplus 1 \sim_h 1 \oplus 1$ in $\mathcal{U}(M_2(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2))$ by [LLR00, Exercice 8.11].

(ii) Let $\sigma \in \mathcal{U}(\mathcal{T}_2)$ be the symmetry $\sigma = s_1s_2^* + s_2s_1^* + p_3$, let $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ be a unitary such that $\jmath_1(s_1s_1^*) = v\jmath_0(s_1s_1^*)v^*$ (Proposition 4.6.(iv)) and set $z := v^*\jmath_1(\sigma)v\jmath_0(\sigma)$.

Then, $q_1 = \jmath_0(s_1s_1^*)$, $q_2 = \jmath_0(s_2s_2^*)$ and $q_3 = z\jmath_0(s_2s_2^*)z^*$ are three properly infinite full projections in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ which satisfy:

– $q_1q_3 = \jmath_0(s_1s_1^*)\,v^*\,\jmath_1(s_2s_2^*)\,v = v^*\,\jmath_1(s_1s_1^*)\jmath_1(s_2s_2^*)\,v = 0 = q_1q_2$,
– $q_2 \sim_h q_1 \sim_h q_3$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ since $\sigma \in \mathcal{U}^0(\mathcal{T}_2)$ and so $z \sim_h v^*v = 1$ in $\mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$,
– $q_1 + q_3 = v^*\jmath_1(s_1s_1^* + s_2s_2^*)v \not\sim_u \jmath_0(s_1s_1^* + s_2s_2^*) = q_1 + q_2$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ by Proposition 4.6.(iii).

Addendum

(iii) Let $\alpha = \alpha_0 * \alpha_1$ be the unital $*$-endomorphism of the free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ defined by $\alpha_0(s_k) = \jmath_0(s_k)$ and $\alpha_1(s_k) = v^*\jmath_1(s_k)$ for $k = 1, 2$. Then $\alpha_0(s_2s_2^* + p_3) = 1 - \alpha_0(s_1s_1^*) = 1 - \alpha_1(s_1s_1^*) = \alpha_1(s_2s_2^* + p_3)$ and $\alpha_0(s_2s_2^*) \sim_h \alpha_0(s_1s_1^*) = \alpha_1(s_1s_1^*) \sim_h \alpha_1(s_2s_2^*)$ among the projections in $\alpha(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$. But $\alpha_0(p_3) \not\sim \alpha_1(p_3)$ in $\alpha(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$.

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